NOTES ON SPLITTING FIELDS

CİHAN BAHRAN

I will try to define the notion of a splitting field of an algebra over a field using my words, to understand it better. The sources I use are Peter Webb's and T.Y Lam's books. Throughout this document, k is a field and A is a (not necessarily finite-dimensional, I will state this assumption when necessary) k-algebra.

The small amount of theory of finite-dimensional algebras I know, like the correspondence between the simple and indecomposable projective modules can be seen to follow from the theory of (left or right) artinian rings. Really, this theory does not make use of the underlying field. In specific examples though, it is quite handy to have k-dimensions of everything around. One way to make use of the underlying field that won't be applicable for general artinian rings is to consider its extensions and extend the scalars.

There we go. Let $k \subseteq F$ be a field extension. Then we naturally have a restriction functor $F\text{-}\mathsf{Alg} \to k\text{-}\mathsf{Alg}$ which has the left adjoint $F \otimes_k - : k\text{-}\mathsf{Alg} \to F\text{-}\mathsf{Alg}$. The unit of this adjunction on A is the k-algebra homomorphism

$$\begin{aligned} \varphi : A &\to F \otimes_k A \\ a &\mapsto 1 \otimes a \end{aligned}$$

Note that φ is injective. Now φ yields a restriction functor $\varphi_* : (F \otimes_k A)\text{-}\mathsf{Mod} \to A\text{-}\mathsf{Mod}$. φ_* has a left adjoint, say $\varphi^* : A\text{-}\mathsf{Mod} \to (F \otimes_k A)\text{-}\mathsf{Mod}$.

Proposition 1. *φ^* is naturally isomorphic to $F \otimes_k -$.*

Proof. The functor $F \otimes_k -$ here cannot be the same with the above, that was between algebra categories. So let's spell out what it is. Note that both A and F are k-algebras. Hence if we have a (left) A-module M and an F-module V, then $V \otimes_k M$ has a natural $F \otimes_k A$-module structure. Taking $V = F$, we see that $F \otimes_k M$ is a $F \otimes_k M$-module. So the assignment

$$\begin{aligned} F \otimes_k - : A\text{-}\mathsf{Mod} &\to (F \otimes_k A)\text{-}\mathsf{Mod} \\ M &\mapsto F \otimes_k M \end{aligned}$$

is well-defined on objects. For morphisms, there is a natural candidate: if $f : M \to M'$ is an A-module map, then let $(F \otimes_k -)(f) = \mathrm{id}_F \otimes f : F \otimes_k M \to F \otimes_k M'$. We have

$$\begin{aligned} (\mathrm{id}_F \otimes f)\left((\alpha \otimes r) \cdot (\beta \otimes m)\right) &= (\mathrm{id}_F \otimes f)(\alpha\beta \otimes rm) \\ &= \alpha\beta \otimes f(rm) \\ &= \alpha\beta \otimes r f(m) \\ &= (\alpha \otimes r) \cdot (\beta \otimes f(m)) \\ &= (\alpha \otimes r) \cdot (\mathrm{id}_F \otimes f)(\beta \otimes m) \end{aligned}$$

for any $\alpha, \beta \in F$, $r \in A$, $m \in M$. Si $\mathrm{id}_F \otimes f$ is really an $F \otimes_k A$-module homomorphism. So we have well-defined assignments on both the objects and morphisms for $F \otimes_k -$ which satisfy the functor axioms.

Now let's move on to the adjunction with φ_*. Given an A-module M and a $F \otimes_k A$-module N, we seek an isomorphism

$$\mathrm{Hom}_A(M, \varphi_*(N)) \cong \mathrm{Hom}_{F\otimes_k A}(F \otimes_k M, N)$$

that is natural in M and N. (Note that N is an A-module via $\varphi : r \mapsto 1_F \otimes r$)
We already have a natural isomorphism

$$\begin{aligned}\eta : \mathrm{Hom}_k(M, N) &\to \mathrm{Hom}_F(F \otimes_k M, N)\\ f &\mapsto \eta(f) : (\alpha \otimes m) \mapsto \alpha f(m) = (\alpha \otimes 1_A) f(m)\end{aligned}$$

by the usual theory of extension of scalars since N is an F-vector space via $\alpha n = (\alpha \otimes 1_A)n$. We need to check that if f is an A-module map then $\eta(f)$ is an $F \otimes_k A$-module map. Indeed, the routine check

$$\begin{aligned}\eta(f)((\beta \otimes r) \cdot (\alpha \otimes m)) &= \eta(f)(\beta\alpha \otimes rm)\\ &= (\beta\alpha \otimes 1_A) f(rm)\\ &= (\beta\alpha \otimes 1_A)(1_F \otimes r) f(m)\\ &= (\beta\alpha \otimes r) f(m)\\ &= (\beta \otimes r)(\alpha \otimes 1_A) f(m)\\ &= (\beta \otimes r) \cdot \eta(f)(\alpha \otimes m)\end{aligned}$$

verifies so. The inverse of η is given by

$$\begin{aligned}\theta : \mathrm{Hom}_F(F \otimes_k M, N) &\to \mathrm{Hom}_k(M, N)\\ g &\mapsto \theta(g) : m \mapsto g(1_F \otimes m)\end{aligned}$$

And if g is an $F \otimes_k A$-module homomorphism, then we have

$$\begin{aligned}\theta(g)(rm) &= g(1_F \otimes rm)\\ &= g((1_F \otimes r) \cdot (1_F \otimes m))\\ &= (1_F \otimes r) \cdot g(1_F \otimes m)\\ &= r \cdot g(1_F \otimes m)\\ &= r \cdot \theta(g)(m)\,,\end{aligned}$$

that is, $\theta(g)$ is an A-module homomorphism. Thus η and θ restrict to the desired isomorphism and they are already natural in M and N. □

Let's take a step back and look at what we've done (gosh I'm bad at writing sometimes). Everything stemmed from the field extension $k \subseteq F$ which gives us a way to induce k-algebras to F-algebras and to induce modules over a fixed k-algebra A to modules over the induced F-algebra. We called this induction functor φ^* above and saw that it is naturally isomorphic to $F \otimes_k -$. I find it more suggestive to write $(-)^F$ for this functor and also A^F for the induced algebra $F \otimes_k A$.

Remark 1. Let M be an A-module such that $\dim_k M$ is finite. Then $\dim_F M^F = \dim_k M$. In particular, $M = 0$ if and only if $M^F = 0$.

Proposition 2. *Let M be an A-module and $k \subseteq F$ a field extension. Consider the A^F-module M^F. If M^F is an indecomposable (simple) A^F-module, then M is an indecomposable (simple) A-module.*

Proof. Better to use contrapositives. By the remark above, a nontrivial decomposition of M into direct summands yields a nontrivial decomposition of M^F. And if N is a nontrivial submodule of M, then N^F embeds into M^F as a submodule since $(-)^F \cong F \otimes_k -$ is an exact functor. N^F is a nontrivial submodule because it is neither 0 nor M^F as $N \neq 0$ and $M/N \neq 0$ so that $N^F \neq 0$ and $0 \neq (M/N)^F \cong M^F/N^F$. □

Let M and N be A-modules. Via the functor $(-)^F : A\text{-}\mathsf{Mod} \to A^F\text{-}\mathsf{Mod}$, we have a map

$$\mathrm{Hom}_A(M,N) \to \mathrm{Hom}_{A^F}(M^F, N^F)\,.$$

This map is actually k-linear. Since the right hand side is an F-vector space, via the adjunction of $(-)^F$ with the restriction, we get an F-linear map

$$\theta : (\mathrm{Hom}_A(M,N))^F \to \mathrm{Hom}_{A^F}(M^F, N^F)\,.$$

Lemma 3. *θ is injective and if moreover $\dim_k M < \infty$, then θ is an isomorphism.*

Proof. Let $\{\alpha_i : i \in I\}$ be a basis of F over k. Note that an arbitrary element of $(\mathrm{Hom}_A(M,N))^F = F \otimes_k \mathrm{Hom}_A(M,N)$ is of the form

$$\sum \alpha_i \otimes f_i$$

where $\alpha_i \in F$ and $f_i : M \to N$ is A-linear. If such an element is in $\ker\theta$, then

$$\begin{aligned} 0 &= \sum \alpha_i (f_i)^F \\ &= \sum \alpha_i (\mathrm{id}_F \otimes f_i)\,. \end{aligned}$$

So for every $m \in M$, we have

$$\begin{aligned} 0 &= \left(\sum \alpha_i (\mathrm{id}_F \otimes f_i)\right)(1 \otimes m) \\ &= \sum \alpha_i \,(1 \otimes f_i(m)) \\ &= \sum \alpha_i \otimes f_i(m)\,. \end{aligned}$$

This equality happens inside N^F and we have

$$\begin{aligned} N^F &= F \otimes_k N \\ &\cong \left(\bigoplus_{i\in I} k\right) \otimes_k N \\ &\cong \bigoplus_{i\in I} k \otimes_k N \\ &\cong \bigoplus_{i\in I} N \end{aligned}$$

where the isomorphism from the bottom row to the top row is given by

$$(n_i)_{i\in I} \mapsto \sum_{i\in I} \alpha_i \otimes n_i\,.$$

The sum makes sense because only finitely many n_i's are nonzero. As a consequence N^F has the decomposition $N^F = \bigoplus_{i\in I} (\alpha_i \otimes_k N)$ as a k-vector space. Hence the above sum being zero implies that $f_i(m) = 0$ for every $i \in I$. m was arbitrary, so f_i's are zero and hence $\sum \alpha_i \otimes f_i = 0$.

Now let $g : M^F \to N^F$ be an A^F-linear map. Since $N^F = \bigoplus_{i\in I} (\alpha_i \otimes_k N)$, for $m \in M$ we have

$$g(1 \otimes m) = \sum \alpha_i \otimes g_i(m)$$

where each g_i is a function from M to N. This representation of $g(1 \otimes m)$ is unique. Therefore via

$$\begin{aligned}\sum \alpha_i \otimes g_i(m + m') &= g(1 \otimes (m + m')) \\ &= g(1 \otimes m + 1 \otimes m') \\ &= g(1 \otimes m) + g(1 \otimes m') \\ &= \sum \alpha_i \otimes g_i(m) + \sum \alpha_i \otimes g_i(m') \\ &= \sum \alpha_i \otimes (g_i(m) + g_i(m')),\end{aligned}$$

we conclude that g_i's are additive maps. For $r \in A$, we have

$$\begin{aligned}g(1 \otimes rm) &= g((1 \otimes r) \cdot (1 \otimes m)) \\ &= (1 \otimes r) \cdot g(1 \otimes m)\end{aligned}$$

since $1 \otimes r \in F \otimes_k A = A^F$ and g is A^F-linear. From here we get

$$\begin{aligned}\sum \alpha_i \otimes g_i(rm) &= (1 \otimes r) \cdot \left(\sum \alpha_i \otimes g_i(m)\right) \\ &= \sum \alpha_i \otimes rg_i(m),\end{aligned}$$

thus g_i's are actually A-linear. Let η be the following composition of k-linear maps

$$M \longrightarrow M^F \xrightarrow{g} N^F \cong \bigoplus_{i\in I} N$$

where the first arrow is the natural map $m \mapsto 1 \otimes m$. By our description, g_i's are precisely the coordinate maps of η. Therefore if $\dim_k M < \infty$, η has finite k-rank and hence only finitely many g_i's can be nonzero. Thus $f = \sum \alpha_i \otimes g_i$ is a legitimate element of $(\mathrm{Hom}_A(M, N))^F$. Now since

$$\theta(f)(1 \otimes m) = \sum \alpha_i \otimes g_i(m) = g(1 \otimes m)$$

for all m and $\theta(f)$, g are F-linear maps, we have $\theta(f) = g$. So θ is surjective. □

Corollary 4. *If M is a finite-dimensional A-module and $k \subseteq F$ is a field extension, then $(\mathrm{End}_A(M))^F \cong \mathrm{End}_{A^F}(M^F)$ as F-algebras.*

Proof. Take $N = M$ in Lemma 3 and observe that the map θ preserves multiplication. □

It is natural to question to what extent the converse of Proposition 2 holds. That is, when does a simple A-module remain simple when induced over extensions of k? The following proposition answers this:

Theorem 5. *Let M be a finite-dimensional simple A-module. TFAE:*

(1) For every field extension $k \subseteq F$, M^F is a simple A^F-module.
(2) If F is an algebraically closed field containing k, then M^F is a simple A^F-module.
(3) $\operatorname{End}_A(M) \cong k$.
(4) The k-algebra homomorphism $A \to \operatorname{End}_k(M)$ is surjective.

Proof. (1) $\Rightarrow$ (2) is trivial. For (2) $\Rightarrow$ (3), we have

$$(\operatorname{End}_A(M))^F \cong \operatorname{End}_{A^F}(M^F) \cong F$$

where the first isomorphism is Corollary 4 and the second isomorphism is by Schur's lemma. Thus

$$1 = \dim_F(\operatorname{End}_A(M))^F = \dim_k \operatorname{End}_A(M)$$

and we deduce that $\operatorname{End}_A(M) \cong k$.

For (3) $\Rightarrow$ (4), write $\varphi : A \to \operatorname{End}_k(M) = E$, $B = \varphi(A)$ and $D = \operatorname{End}_A(M)$. We want to show $B = E$. The assumption is that every element in D is a scalar multiplication, so $D = \mathbf{Z}(E)$ - the center of the endomorphism algebra. Also by definition we have $D = \mathbf{C}_E(B)$. Now B is a finite-dimensional k-algebra, hence left (and right) Artinian. Also M is a simple B-module, therefore by the double-centralizer theorem

$$B = \mathbf{C}_E(D) = \mathbf{C}_E(\mathbf{Z}(E)) = E\,.$$

For (4) $\Rightarrow$ (1), write $E = \operatorname{End}_k(M)$. Now M is the only simple module of E since picking any k-basis $e_1, \ldots, e_n$ for M establishes isomorphisms $M \cong k^n$ and $E \cong \mathbb{M}_n(k)$ where the action is matrix multiplication. Note that for a field extension $k \subseteq F$, we have $M^F \cong F^n$ and $E^F \cong \mathbb{M}_n(F)$ where the E^F action on M^F is again matrix multiplication. Thus M^F is simple as an E^F-module.

Applying $(-)^F$ to the surjection $A \to E$ yields an F-algebra map

$$A^F \to E^F \cong \operatorname{End}_F(M^F)$$

where the isomorphism is by Corollary 4. This surjective ($(-)^F$ is exact!) map is the one that gives the A^F-module structure on M^F, so the E^F-submodules and A^F-submodules of M^F coincide. Thus M^F is a simple A^F-module. □

From now on, we assume that A is a finite-dimensional k-algebra.

We call a simple A-module - which is necessarily finite-dimensional since it is a quotient of ${}_AA$ - satisfying one (and hence all) of the conditions in Theorem 5 **absolutely simple**. Note that if S is an absolutely simple n-dimensional A-module, its multiplicity in $A/\operatorname{Rad} A$ is n and the matrix algebra corresponding to S in the Artin-Wedderburn decomposition of $A/\operatorname{Rad} A$ is $\mathbb{M}_n(k)$.

Example. Let G be a finite group. Then the trivial module k is always an absolutely simple kG-module. More generally every 1-dimensional module S is absolutely simple because $\operatorname{End}_k(S)$ is already isomorphic to k, hence so is $\operatorname{End}_{kG}(S)$.

We call A a **split** k-algebra if every simple A-module is absolutely simple. By our observation above, we have the following fact:

Proposition 6. *Suppose A is split. Then if $\mathcal{S}$ is a complete list of non-isomorphic simple A-modules, then*

$$A/\operatorname{Rad} A \cong \bigoplus_{S \in \mathcal{S}} S^{\dim S}$$

and so

$$\dim(A/\operatorname{Rad} A) = \sum_{S\in\mathcal{S}} (\dim S)^2\,.$$

Example. $\mathbb{M}_n(k)$ is a split k-algebra. We showed this in the proof of (4) $\Rightarrow$ (1) in Theorem 5.

We call an extension field F of k a **splitting field** for A if A^F is a split F-algebra.

Observe that by (2) in Theorem 5, if k is algebraically closed then A is split. More generally, if F is an algebraically closed field containing k then F is a splitting field for A. So splitting fields always exist as we can go to the algebraic closure $\overline{k}$. But $\overline{k}$ is huge and really an overkill - note that for a group algebra having some roots of unity around may be enough. Actually splitting fields can always be found in finite extensions of k, which is our next aim to show.

If A is split, the extension of scalars functor has nicer properties. Note that given a field extension $k \subseteq F$, by applying the exact functor $(-)^F$ to the inclusion $\operatorname{Rad} A \hookrightarrow A$ we may identify $(\operatorname{Rad} A)^F$ as a submodule (actually a two-sided ideal) of A^F.

Proposition 7. *With this identification,* $(\operatorname{Rad} A)^F = \operatorname{Rad} A^F$ *when A is a split k-algebra.*

Proof. The (left or right) A^F-module $A^F/(\operatorname{Rad} A)^F \cong (A/\operatorname{Rad} A)^F$ is semisimple since A is split. Hence $\operatorname{Rad} A^F \subseteq (\operatorname{Rad} A)^F$. On the other hand, $(\operatorname{Rad} A)^F = F \otimes_k \operatorname{Rad} A$ is a nilpotent ideal of $A^F = F \otimes_k A$ so we get the reverse inclusion. □

We can do this for modules too. Note that similar to above, given an A-module U and a submodule $V \subseteq U$, we can identify V^F as an A^F-submodule of U^F.

Corollary 8. *Suppose A is a split k-algebra. Let U be a finite-dimensional A-module. Then identifying* $(\operatorname{Rad} U)^F$ *as a submodule of U^F, we have* $(\operatorname{Rad} U)^F = \operatorname{Rad} U^F$.

Proof. We have

$$\begin{aligned}
\operatorname{Rad} U^F &= (\operatorname{Rad} A^F)\cdot U^F \\
&= (\operatorname{Rad} A)^F \cdot U^F \\
&= (F\otimes_k \operatorname{Rad} A)\cdot (F\otimes_k U) \\
&= F\otimes_k (\operatorname{Rad} A\cdot U) \\
&= F\otimes_k \operatorname{Rad} U \\
&= (\operatorname{Rad} U)^F
\end{aligned}$$

where the first and fifth equality holds because U^F is an artinian A^F-module and U is an artinian A-module. □

The next lemma gives us a way to check whether a module is a result of extension of scalars:

Lemma 9. *Let $k \subseteq F$ be a field extension and V be a finite-dimensional A^F-module. Then $V \cong U^F$ for some A-module U if and only if V has an F-basis $v_1, \ldots, v_n$ such that when $\operatorname{End}_F(V)$ is identified with $\mathbb{M}_n(F)$ using this basis, the image of the composite map $A \to A^F \to \operatorname{End}_F(V)$ lies in $\mathbb{M}_n(k)$.*

Proof. Assume $V \cong U^F$ for some U. Since $\dim_k U = \dim_F V < \infty$, we can pick a k-basis $u_1, \dots u_n$ for U and get a k-algebra isomorphism $\mathrm{End}_k(U) \cong \mathbb{M}_n(k)$ out of it. Since $(-)^F : k\text{-}\mathsf{Alg} \to F\text{-}\mathsf{Alg}$ is the left adjoint of the restriction functor, we have a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & \mathrm{End}_k(U) \\ \downarrow & & \downarrow \\ A^F & \longrightarrow & (\mathrm{End}_k(U))^F \cong \mathrm{End}_F(V) \end{array}$$

of k-algebras using the unit of the adjunction. Note that $1 \otimes u_i$'s are an F-basis for V which gives an isomorphism $\mathrm{End}_F(V) \cong \mathbb{M}_n(F)$ and the map which completes

$$\begin{array}{ccc} \mathrm{End}_k(U) & \longrightarrow & \mathbb{M}_n(k) \\ \downarrow & & \\ \mathrm{End}_F(V) & \longrightarrow & \mathbb{M}_n(F) \end{array}$$

into a commutative square is just the inclusion $\mathbb{M}_n(k) \to \mathbb{M}_n(F)$. Thus the image of the composite map $A \to A^F \to \mathrm{End}_F(V)$ lies in $\mathbb{M}_n(k)$ by the commutativity of the first diagram.

Conversely, assume V has such a basis $v_1, \dots, v_n$. Let U be the k-span of these vectors inside V. Then by assumption, U is an A-submodule of V. Now a pure tensor $\alpha \otimes r \in F \otimes_k A = A^F$ acts on a basis element v_i by

$$(\alpha \otimes r) \cdot v_i = (\alpha \cdot (1 \otimes r)) \cdot v_i = \alpha \cdot (r v_i)\,.$$

Note that $r v_i \in U$ here. On the other hand, U^F has an F-basis consisting of $1 \otimes u_i$'s and we have

$$(\alpha \otimes r) \cdot (1 \otimes u_i) = (\alpha \otimes r u_i) = \alpha \cdot (1 \otimes r u_i) = \alpha \cdot (r(1 \otimes u_i))$$

so the bijection $v_i \leftrightarrow 1 \otimes u_i$ preserves the A^F-action and $V \cong U^F$. □

When an A^F-module V turns out to be an induced from an A-module as in Lemma 9, we say that V **can be written in** k.

Theorem 10. *Let $k \subseteq E \subseteq F$ be field extensions. TFAE:*

(1) F is a splitting field for A and every simple A^F-module can be written in E.
(2) E is a splitting field for A.

Proof. (1) $\Rightarrow$ (2): Let $\mathcal{V}$ to be a complete list of non-isomorphic simple (and hence absolutely simple) A^F-modules. Then there is a list, say $\mathcal{U}$, of A^E-modules such that every $V \in \mathcal{V}$ is isomorphic to U^F for some $U \in \mathcal{U}$.

Note that every $U \in \mathcal{U}$ is simple by Proposition 2. Now let K be an algebraically closed field containing F (we may pick $K = \overline{F}$ for instance). Then for every $U \in \mathcal{U}$, the A^K-module $U^K \cong (U^F)^K$ is simple since U^F is absolutely simple.

So by part (2) of Theorem 5 every $U \in \mathcal{U}$ is absolutely simple. We want to show that $\mathcal{U}$ is a complete list of simple A^E-modules (up to isomorphism) to deduce that E is a splitting field for A.

To that end, let S be a simple A^E-module. Then there exists an idempotent $e \in A^E$ such that $eS \neq 0$ but e annihilates every other simple A^E-module (see Theorem 7.13 in Webb's book). Since A^E embeds into A^F we can consider e as an idempotent in A^F. Since $\operatorname{Rad} A^F$ does not contain idempotents, there exists $V \in \mathcal{V}$ such that $eV \neq 0$. Now pick $U \in \mathcal{U}$ such that $V \cong U^F$. Then e cannot annihilate U, which forces $U \cong S$.

(2) ⇒ (1): We can replace E with k and assume that A is already split. Let $\mathcal{S}$ be a complete list of non-isomorphic simple A-modules. So we have an A-module isomorphism

$$A/\operatorname{Rad} A \cong \bigoplus_{S\in\mathcal{S}} S^{n_S}$$

for some n_S (we actually know by Proposition 6 that $n_S = \dim S$ but we don't need this fact here). Applying $(-)^F$ here and using Proposition 7, we get an A^F-module isomorphism

$$A^F/\operatorname{Rad} A^F \cong \bigoplus_{S\in\mathcal{S}} \left(S^F\right)^{n_S} .$$

Note that each S^F is a simple A^F-module because each S is absolutely simple. Thus every simple A^F-module must be isomorphic to some S^F. In other words, simple A^F-modules can be written in k.

It remains to show that F is a splitting field for A, that is, A^F is a split F-algebra. So let U be a simple A^F-module and $F \subseteq K$ a field extension of F. By what we just showed, $U \cong S^F$ for some simple A-module S. Since A is split, $S^K = (S^F)^K \cong U^K$ is a simple $A^K = (A^F)^K$-module. □

Proposition 11. *Let F be an algebraic extension of k and V a finite-dimensional A^F-module. Then there exists an intermediate field $k \subseteq E \subseteq F$ with $[E : k] < \infty$ such that V can be written in E.*

Proof. Let $v_1, \dots, v_n$ be an F-basis of V with which we can identify $\operatorname{End}_F(V)$ with $\mathbb{M}_n(F)$. Pick a k-basis $a_1, \dots, a_t$ for A and let $B_1, \dots, B_t$ be their images under the composite map $A \to A^F \to \mathbb{M}_n(F)$. Let E be the subfield of F generated by k and the entries of B_i's. Since E is generated over k by finitely many algebraic elements, $k \subseteq E$ is a finite extension and by construction B_i's lie in $\mathbb{M}_n(E)$. The a_i's gets sent to an E-basis of A^E under the natural map $A \to A^E$ and hence the composite $A^E \to (A^E)^F = A^F \to \mathbb{M}_n(F)$ has image contained in $\mathbb{M}_n(E)$. Invoke Lemma 9. □

Corollary 12. *A has a splitting field which has finite degree over k.*

Proof. Let $F = \overline{k}$. Being a finite-dimensional F-algebra, A^F has finitely many simple modules. So by repeated application of Proposition 11 (the extension $k \subseteq F$ is algebraic!), we get an intermediate field $k \subseteq E \subseteq F$ with $[E : k] < \infty$ such that every simple A^F-module can be written in E. Theorem 10 applies and E is a splitting field for A. □

Corollary 12 shows that there are relatively small splitting fields for A. On the other extreme, we might want to get large splitting fields. A natural question would be that if F_1 and F_2 are splitting fields for A, can we find a larger splitting field which contains both F_i? By Theorem 10, field extensions of splitting fields are splitting fields themselves, so the following lemma suffices for a positive answer.

Lemma 13. *Let F_1 and F_2 be field extensions of k. Then there is a field extension F of k such that there are injective k-algebra maps $F_1 \hookrightarrow F$ and $F_2 \hookrightarrow F$.*

Proof. From the k-algebra $B = F_1 \otimes_k F_2$. Since $B \neq 0$, it has a maximal ideal $\mathfrak{m}$. Then the k-algebra $F = B/\mathfrak{m}$ is a field and the composite k-algebra map $F_i \to F \twoheadrightarrow F$ is necessarily injective as F_i is a field. □