ALPERIN'S FUSION THEOREM

CİHAN BAHRAN

I will try to reproduce the proof of Alperin's fusion theorem for a saturated fusion system. The sources I use are AKO(Aschbacher, Kessar and Oliver) and Craven. Throughout this document, $\mathcal{F}$ is a saturated fusion system over a finite p-group $S \neq 1$.

Recall that given an isomorphism $\varphi : P \to Q$ in $\mathcal{F}$, $g \in N_\varphi$ if and only if $g \in N_S(P)$ and there exists $h \in N_S(Q)$ such that the diagram

$$\begin{tikzcd} P \arrow[r, "c_g"] \arrow[d, "\varphi"'] & P \arrow[d, "\varphi"] \\ Q \arrow[r, "c_h"] & Q \end{tikzcd}$$

commutes. The subgroup $N_\varphi \leq N_S(P)$ has the property that if $\varphi : P \to Q$ extends to $\overline{\varphi} : R \to S$ in $\mathcal{F}$ for some $R \leq N_S(P)$ then $R \leq N_\varphi$. And φ extends in S if and only if extends in $N_S(P)$: indeed if φ extends to $U > P$ for some $U \leq S$, then it extends to $N_U(P) > P$, which is contained in $N_S(P)$.

Definition 1. An isomorphism $\varphi : P \to Q$ in $\mathcal{F}$ is called **$\mathcal{F}$-domestic** if $N_\varphi = P$. A subgroup P is called a **$\mathcal{F}$-domestic intersection** if P is fully $\mathcal{F}$-normalized and there exists a $\mathcal{F}$-domestic isomorphism $\varphi : P \to Q$ with Q also fully $\mathcal{F}$-normalized.

So domestic isomorphisms in $\mathcal{F}$ are precisely those that cannot be extended (not just in $\mathcal{F}$ but as a group homomorphism) to a larger subgroup of S than their domain.

In case $\mathcal{F}$ is the fusion system coming from a finite group, domestic intersections are tame intersections (which appear in the original statement of Alperin's Fusion Theorem).

Proposition 2. *If G is a finite group with $S \in Syl_p(G)$ and $\mathcal{F} = \mathcal{F}_G(S)$, then an $\mathcal{F}$-domestic intersection P in $\mathcal{F}$ is a tame intersection in G.*

Proof. Since $\mathcal{F} = \mathcal{F}_G(S)$, the definition above yields that there exists $g \in G$ such that ${}^gP \leq S$ and both P and ${}^gP$ are fully normalized, so $N_S(P) \in \mathrm{Syl}_p(N_G(P))$ and $N_S({}^gP) \in \mathrm{Syl}_p(N_G({}^gP))$.

So we have

- $P \leq S \cap S^g$,
- $N_S(P) \in \mathrm{Syl}_p(N_G(P))$,
- $N_{S^g}(P) \in \mathrm{Syl}_p(N_G(P))$.

And the fact that the conjugation $c_g : P \to {}^gP$ can be extended to $S \cap S^g \to {}^gS \cap S$ in $\mathcal{F}$ forces $P = S \cap S^g$. Hence P is a tame intersection. $\square$

Let us recall some facts about groups with strongly p-embedded subgroups.

Definition 3. Let G be a finite group. A proper subgroup $H < G$ is called **strongly p-embedded** in G if $p \mid |H|$ but for every $x \in G - H$ we have $p \nmid |{}^xH \cap H|$.

Proposition 4. *Let G be a finite group and H be a strongly p-embedded subgroup of G. Then H contains a Sylow p-subgroup of G.*

Proof. Since $p \mid |H|$, there exists $g \in H$ of order p. Now let $T \in \mathrm{Syl}_p(G)$ such that $g \in T$. As $p \mid |G|$, T is nontrivial, hence so is $Z(T)$. Hence there exists $x \in Z(T)$ of order p. Now $g^x = g \in H$, so $g \in {}^xH \cap H$. Therefore $p \mid |{}^xH \cap H|$ so by definition of being strongly p-embedded we get $x \in H$. Now for every $t \in T$, we have $x^t = x \in H$, so $x \in {}^tH \cap H$ and similarly this yields $t \in H$. Thus $T \subseteq H$. $\square$

Corollary 5. *Let G be a finite group such that $S \in Syl_p(G)$. If the set*

$$\mathcal{H} = \{H \leq G : H \text{ is strongly } p\text{-embedded in } G\}$$

is non-empty, then there exists $H \in \mathcal{H}$ which contains S.

Proof. Pick $K \in \mathcal{H}$. Then we know that K contains a Sylow p-subgroup of G, which is necessarily of the form S^g. Then $H := {}^gK \in \mathcal{H}$ and $S \subseteq H$. $\square$

Corollary 6. *If a finite group G contains a strongly p-embedded subgroup, then $O_p(G) = 1$.*

Proof. Let H be a strongly p-embedded subgroup of G. Then there exists $T \in \mathrm{Syl}_p(G)$ such that $T \subseteq H$. By definition $H < G$ so we may pick $x \in G - H$ for which ${}^xH \cap H$ must be a p'-group. But on the other hand ${}^xT \cap T$ is a p-subgroup of ${}^xH \cap H$ so ${}^xT \cap T = 1$. But $O_p(G)$ is the intersection of the Sylow p-subgroups of G, so $O_p(G) = 1$. $\square$

Now by using the notion of strongly p-embedded we define a class of subgroups in $\mathcal{F}$ which turn out to be domestic intersections.

Definition 7. $P \leq S$ is called **$\mathcal{F}$-essential** if P is fully normalized, $\mathcal{F}$-centric and $\mathrm{Out}_{\mathcal{F}}(P) = \mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Inn}(P)$ contains a strongly p-embedded subgroup.

Proposition 8. *Essential subgroups are domestic intersections.*

Proof. Let P be an essential subgroup. Since P is fully normalized and $\mathcal{F}$ is saturated, P is fully automized, that is, $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$. So $W := \mathrm{Aut}_S(P)/\mathrm{Inn}(P)$ is a Sylow p-subgroup of $\mathrm{Out}_{\mathcal{F}}(P)$ and hence $\mathrm{Out}_{\mathcal{F}}(P)$ has a strongly p-embedded subgroup M containing W. Pick $\theta \in \mathrm{Out}_{\mathcal{F}}(P) - M$, then $M^\theta \cap M$ is a p'-group so $W \nsubseteq M^\theta$, or equivalently ${}^\theta W \nsubseteq M$. Moreover, since ${}^\theta W \cap W$ is a p-subgroup of the p'-group ${}^\theta M \cap M$ we have ${}^\theta W \cap W = 1 = W \cap W^\theta$.

Write $\theta = \overline{\varphi}$ where $\varphi \in \mathrm{Aut}_{\mathcal{F}}(P)$. Then we have

$$\mathrm{Aut}_S(P) \cap \mathrm{Aut}_S(P)^\varphi = \mathrm{Inn}(P)\,,$$

and since N_φ is precisely the inverse image of this subgroup of $\mathrm{Aut}_S(P)$ under the surjection $N_S(P) \to \mathrm{Aut}_S(P)$, we get $N_\varphi = P$. So φ is domestic and both P and $\varphi(P) = P$ are *a priori* fully normalized so P is a domestic intersection. $\square$

We are almost ready to prove Alperin's fusion theorem. Let's prove another proposition about strongly p-embedded subgroups.

Proposition 9 (AKO)**.** *Let G be a finite group and $1 \neq T \in Syl_p(G)$. Set*

$$H_T = \langle x \in G : {}^xT \cap T \neq 1 \rangle\,.$$

(1) If $H_T = G$ then G contains no strongly p-embedded subgroups.

(2) If $H_T < G$ then H_T is a strongly p-embedded subgroup of G. Moreover if H is any strongly p-embedded subgroup of G that contains T, then H also contains H_T.

Proof. (1) We show the contrapositive. Assume that G contains a strongly p-embedded subgroup. Then G contains a strongly p-embedded subgroup H that contains T. But then for every $x \in G - H$ we have ${}^xT \cap T = 1$. Equivalently, ${}^xT \cap T \neq 1$ implies $x \in H$. Thus $H_T \leq H < G$.

(2) We showed the last part of (2) in (1). It remains to show that if $H_T < G$ then H_T is strongly p-embedded in G. Assume $x \in G$ such that $p \mid |{}^xH_T \cap H_T|$. We want to show that $x \in H_T$. Pick $g \in {}^xH_T \cap H_T$ of order p. Note that $T \leq H_T$, so $T \in \mathrm{Syl}_p(H_T)$. And since g^x and g are elements of order p in H_T, there exists $y, z \in H_T$ such that $g^{xy} \in T$ and $g^z \in T$. So

$$g^{xy} = (g^z)^{z^{-1}xy} \in T \cap T^{z^{-1}xy},$$

hence $z^{-1}xy \in H_T$; so $x \in H_T$.

Also as $T \leq H_T$, $p \mid |H|$. Thus H_T is strongly p-embedded in G. □

Corollary 10. *Let P be a proper subgroup of S that is fully $\mathcal{F}$-normalized. Set*

$$E_P = \langle \alpha \in \mathrm{Aut}_{\mathcal{F}}(P) : \alpha \textit{ is } \textbf{\textit{not}} \ \mathcal{F}\textit{-domestic} \rangle .$$

(1) If $E_P = \mathrm{Aut}_{\mathcal{F}}(P)$, then P is not $\mathcal{F}$-essential.
(2) If $E_P < \mathrm{Aut}_{\mathcal{F}}(P)$, then P is $\mathcal{F}$-essential and $E_P/\mathrm{Inn}(P)$ is a strongly p-embedded subgroup of $\mathrm{Out}_{\mathcal{F}}(P)$.

Proof. We want to apply Proposition 9. Let $G = \mathrm{Out}_{\mathcal{F}}(P)$ and $T = \mathrm{Out}_S(P)$. Since P is fully $\mathcal{F}$-normalized and $\mathcal{F}$ is saturated, P is fully automized and hence $T \in \mathrm{Syl}_p(G)$.

Observe that for every $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$, N_α contains $PC_S(P)$. Hence if P is not $\mathcal{F}$-centric then $N_\alpha > P$ for every α, which gives $E_P = \mathrm{Aut}_{\mathcal{F}}(P)$.

So we may assume P is $\mathcal{F}$-centric. Then $PC_S(P) = P$ and so

$$T \cong N_S(P)/PC_S(P) = N_S(P)/P > 1$$

as $P < S$ and S is a p-group.

Also, conjugation by elements of P evidently extends to the larger group S, so $\mathrm{Inn}(P) \leq E_P$. We claim that in Proposition 9's notation, we have $H_T = E_P/\mathrm{Inn}(P)$. For this we need to show that

$$E_P = \langle \alpha \in \mathrm{Aut}_{\mathcal{F}}(P) : {}^\alpha \mathrm{Aut}_S(P) \cap \mathrm{Aut}_S(P) > \mathrm{Inn}(P) \rangle .$$

But indeed, α is not domestic if and only if $N_\alpha > P$. And considering the surjection $c_P : N_S(P) \to \mathrm{Aut}_S(P)$, this happens if and only if

$$\mathrm{Inn}(P) < c_P(N_\alpha) = \mathrm{Aut}_S(P) \cap \mathrm{Aut}_S(P)^\alpha$$

which is, by taking α-conjugates, what we need.

So if $E_P = \mathrm{Aut}_{\mathcal{F}}(P)$, then $H_T = G$ so by Proposition 9, $G = \mathrm{Out}_{\mathcal{F}}(P)$ does not contain a strongly p-embedded subgroup. Hence P is not $\mathcal{F}$-essential.

And if $E_P < \mathrm{Aut}_{\mathcal{F}}(P)$, then $H_T < G$ so again by Proposition 9, H_T is strongly p-embedded in G. We are already assuming P is fully $\mathcal{F}$-normalized and $\mathcal{F}$-centric, so P is $\mathcal{F}$-essential. □

Definition 11. Given a subset of morphisms $\mathcal{M}$ in $\mathcal{F}$, we write $\overline{\mathcal{M}}$ for the collection of morphisms that are obtained by restricting the morphisms in $\mathcal{M}$ to subgroups.

So if $\varphi : U \to V$ is in $\mathcal{M}$ and $U' \leq U$ such that $\varphi(U') \leq V'$, then the restriction $U' \to V'$ of φ is in $\overline{\mathcal{M}}$. By definition of being a fusion system, we have $\overline{\mathcal{F}} = \mathcal{F}$ so $\overline{\mathcal{M}}$ is also a subset of morphisms in $\mathcal{F}$.

We finally prove a form of Alperin's fusion theorem for fusion systems:

Theorem 12. *Let $\mathcal{M} = \{\varphi \in \mathrm{Aut}_{\mathcal{F}}(P) : P = S$ or P is $\mathcal{F}$-essential$\}$. Then $\overline{\mathcal{M}}$ generates $\mathcal{F}$ as a category.*

Proof. Write $\mathcal{F}'$ for the subcategory that $\overline{\mathcal{M}}$ generates. Suppose $\mathcal{F} \neq \mathcal{F}'$. Then choose φ to be a morphism in $\mathcal{F}$ but not in $\mathcal{F}'$ whose domain has the largest order possible. Say $\varphi : P \to Q$. Note that the only morphisms in $\mathcal{F}$ with domain S are the automorphisms of S which are already in $\mathcal{M}$. So $P < S$.

Now the inclusion $\varphi(P) \hookrightarrow Q$ is in $\overline{\mathcal{M}}$ (it is a restriction of the identity $S \to S$) so the isomorphism $P \to \varphi(P)$ cannot be in $\mathcal{F}'$. Hence we may assume φ to be an isomorphism.

Now let R to be a fully normalized member of the $\mathcal{F}$-isomorphism class $[P] = [Q]$. Since $\mathcal{F}$ is saturated, R is fully automized and receptive, so there exist morphisms $\alpha : N_S(P) \to N_S(R)$ and $\beta : N_S(Q) \to N_S(R)$ in $\mathcal{F}$ such that $\alpha(P) = R$ and $\beta(Q) = R$ (Lemma 2.6 in AKO). Since $P < S$, we have $N_S(P) > P$ and $N_S(Q) > Q$, so in particular $N_S(P)$ and $N_S(Q)$ have larger order than P. Thus by our choice of P, α and β are in $\mathcal{F}'$. Since $\overline{\mathcal{F}'} = \mathcal{F}'$, the isomorphisms $\alpha|_P : P \to R$ and $\beta|_Q : Q \to R$ are also in $\mathcal{F}'$. Thus

$$(\alpha|_P)^{-1} \circ \beta|_Q \circ \varphi \in \mathrm{Aut}_{\mathcal{F}}(P)$$

is not in $\mathcal{F}'$. So we may assume $\varphi \in \mathrm{Aut}_{\mathcal{F}}(P)$.

If P is $\mathcal{F}$-essential then $\varphi \in \mathcal{M}$, a contradiction. If P is not $\mathcal{F}$-essential, then by Corollary 10 we have

$$\mathrm{Aut}_{\mathcal{F}}(P) = \langle \alpha \in \mathrm{Aut}_{\mathcal{F}}(P) : \alpha \text{ is not } \mathcal{F}\text{-domestic} \rangle .$$

So $\varphi = \alpha_1 \cdots \alpha_n$ where each $\alpha_i \in \mathrm{Aut}_{\mathcal{F}}(P)$ is not $\mathcal{F}$-domestic. So each α_i can be extended to a larger group than P, so again by the choice of P this forces α_i's to be in $\mathcal{F}'$. But φ was not in $\mathcal{F}'$; a contradiction. $\square$